Measure Theory

Riley Moriss

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Contents

Chapter 1

Lebesgue Measure

I omit the theorems that will be put into a more general setting later.

Measureable Sets

Cubes

A closed rectangle in $\mathbb{R}^d$ is simply a set of the form

$$R = [a_1, b_1] \times ... \times [a_d, b_d]$$

with volume

$$|R| = (b_1 - a_1) \times ... \times (b_d - a_d)$$

Two rectangles are almost disjoint if their interior are disjoint (agree on a zero set of measure zero).

Lemma. *Given a rectangle R, that is the union of a finite collection of almost pairwise disjoint rectangles $\{R_i\}$ we have that*

$$|R| = \sum |R_i|$$

proof

Moreover if they are not pairwise disjoint we have

$$|R| \leq \sum |R_i|$$

Theorem. *Every open subset U of $\mathbb{R}$can be written uniquely as a countable union of pairwise disjoint open intervals*

proof

Theorem. *Every open subset U of $\mathbb{R}^d$ can be written as the countable union of pairwise almost disjoint closed cubes*

proof

Outer Measure

The Lebesgue outer measure is definied on any subset of $\mathbb{R}^d$ by

$$m^*(E) = \inf \left\{ \sum_{i \in \mathbb{N}} |Q_i| : \{Q_i\} \text{ is a countale cover of closed cubes of } E \right\}$$

Lemma. *For any ϵ there is a covering of E by closed cubes $\{Q_i\}$ such that*

$$\sum |Q_i| \leq m^*(E) + \epsilon$$

This outer measure is monotone. Countable subadditivity.

Cantor Set Let $C_0 = [1, 0]$ then $C_1 = [0, 1/3] \cup [2/3, 1]$ is removing the middle open third interval of C_0. Continue indefinitely. Then the cantor set is

$$C = \cap_k C_k$$

Cantor set is closed, with empty interior. C is totally disconnected. It is uncountable.

$$C = [0, 1] \setminus \bigcup_{n \in \mathbb{N}} \bigcup_{k=0}^{3^n - 1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}}\right) = \bigcap_{n \in \mathbb{N}} \bigcup_{k=0}^{3^n - 1} \left[\frac{3k}{3^n}, \frac{3k+1}{3^n}\right] \cup \left[\frac{3k+2}{3^n}, \frac{3k+3}{3^n}\right]$$

Lemma.

$$m^*(E) = \inf\{m^*(U) : E \subseteq U \; U \textit{ is open } \}$$

Lemma. *If E_1, E_2 are subsets with some non-zero distance between them (inf over the distances between all the points in the set) then*

$$m^*(E_1 \cup E_2) = m^*(E_1) + m^*(E_2)$$

proof

Lemma. *If E is the union of a countable collection of pariwise almost disjoint cubes $\{Q_i\}$ then*

$$m^*(E) = \sum |Q_i|$$

A subset $E \subseteq \mathbb{R}^d$ is Lebesgue measurable iff for every $\epsilon > 0$ there is an open subset $E \subseteq U$ such that

$$m^*(U \setminus E) \leq \epsilon$$

The Lebesgue measure of a Lebesgue measurabel set is defined to be the Lebesgue outer measure of the set.

$$m(E) = m^*(E)$$

Lemma. *Lebesgue measure is complete. i.e. Subsets of sets with measure zero are measurable and have measure zero*

proof

The collection of measurable sets is a $\sigma-$Algebra.

proof

The smallest $\sigma-$Algebra containing all open sets is called the Borel $\sigma-$Algebra.
Lebesgue measure is countably additive.

proof

Approximation of Sets

Lemma. *Every open subset is measurable. Every closed subset is measurable.*

proof

Definition: *Given a countable collection of of subsets of $\mathbb{R}^d$ $\{E_n\}$ then we say*

- *$E_n \nearrow E$ iff $E_n \subseteq E_{n+1}$ and $E = \cup E_n$*
- *$E_n \searrow E$ iff $E_{n+1} \subseteq E_n$ and $E = \cup E_n$*

And the Lebesgue measure is continuous as a general one is.

proof

Recall that

$$E \bigtriangleup F = (E \setminus F) \cup (F \setminus E)$$

Lemma. *For every $\epsilon > 0$ and a measurable set E*

- *There is a closed set F $F \subseteq E$ with $m(E \setminus F) < \epsilon$*
- *If $m(E) < \infty$ then there is a compact set K such that $K \subseteq E$ with $m(E \setminus K) < \epsilon$*
- *If $m(E) < \infty$ then there is a finite unino of closed cubes $F = \cup_{k=1}^{N} Q_k$ such that $m(E \bigtriangleup F) < \epsilon$*

proof

Littlewood Principles

1. Every measurable set is nearly a finite union of intervals
2. Every measurable function is nearly continuous
3. Every convergent sequence of measurable functions is nearly uniformely convergent

We have seen the first earlier.

Theorem. *Let E be measurable with $m(E) < \infty$ and $f : E \to \mathbb{R}$ measurable. Then for any $\epsilon > 0$ there exists a closed set F such that $F \subseteq E$ and $m(E \setminus F) \leq \epsilon$ and*

$$f|F \text{ is continuous}$$

proof

Ergorovs theorem is the third principle.

Non-Measurable Sets

construction of non-measurable sets

Measureable Functions

prrof

Measurable functions $f : \mathbb{R}^d \to \mathbb{R}$ are those where the preimae of galf lines are measurable. It doesnt matter if they are open , closed or upwards or downwards.

proof

Lemma. *f is measurable iff the preimage of opens is measurable iff preimage of closed is measurable*

proof

note that powers, sums, products, quotients (where non-zero), and scalar multiples of measurable functions are all measurable. Moreover (point wise) limits, liminf, limsup, inf and sups of sequences of measurable functions are all measurable.

proof

Lemma. *If f is measurable and $g = f$ a.e. then g is measurable.*

Integration

Approximation of Measurable Functions

All the same as those of the general section. Theorems of linearity, monotonicity, dominated convergence, monotone convergence, integrable implies finite a.e., triagnle inequality, bounded convergence theorem, Riemann and Lebesgue integral match, interchanging sums, Fatou

Lemma (Borel-Cantelli)**.** *$\{E_k\}$ countable collection of measurable sets such that $\sum m(E_k) < \infty$ then*

$$m(\{x : x \in E_k \text{ for infinitely many } k\}) = 0$$

‘

proof

Lemma. *Let f be integrable*

- *For every $\epsilon > 0$ there is a ball B*
$$\int_{\mathbb{R}^d \setminus B} |f| < \epsilon$$
- *(Absolutely Continuity) For every $\epsilon > 0$ there is a $\delta > 0$ such that*
$$m(E) < \delta \implies \int_E |f| < \epsilon$$

proof

Reimann Integral

section on their equality

Chapter 2

General Measures

Definition: *A σ-Algebra on a set X is some $\mathcal{A} \subseteq \mathcal{P}(X)$ such that*

- $\emptyset \in \mathcal{A}$ *and* $X \in \mathcal{A}$
- *Closed under compliments*
$$E \in \mathcal{A} \implies E^C \in \mathcal{A}$$
- *Closed under countable unions*
$$\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A} \implies \mathbb{U}_{n\in\mathbb{N}} E_n \in \mathcal{A}$$

A set and a sigma algebra are known as a measurable space.

Measures

Definition: *Given a measure space $(X, \mathcal{A})$ then a function $\mu : \mathcal{A} \to [0, \infty]$ is a measure iff*

- $\mu(\emptyset) = 0$
- *If $\{E_n\}_{n\in\mathbb{N}}$ are pairwise disjoint sets in $\mathcal{A}$ then*

$$\mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \sum_{n\in\mathbb{N}} \mu(E_n)$$

A measurable space, (X, A), with a measure, μ, is called a measure space (X, A, μ).

Definition: *A measure space (X, A, μ) is sigma finite (σ-finite) if there is a countable collection $\{E_n\} \subseteq A$ such that $X = \cup E_n$ and for every n we have $\mu(E_n) < \infty$.*

Definition: *A measure space (X, A, μ) is complete iff for every $E \in A$ with measure zero we have for every $F \subseteq E$ F is both measurable and $\mu(F) = 0$.*

Every subset of measure zero set is measurable and has measure zero. The Lebesgue measure is complete, the completion of the Borel measure infact.

Measures have the following properties:

- Monotonicity

$$E \subseteq F \implies \mu(E) \leq \mu(F)$$

- Countable Subadditivity

$$\mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) \leq \sum_{n\in\mathbb{N}} \mu(E_n)$$

- Given a collection $\{E_n\}_{n\in\mathbb{N}}$ of measurable sets we have that

$$E_n \nearrow E \implies \mu(E) = \lim_{n\to\infty} \mu(E_n)$$

$$E_n \searrow E \wedge \exists n\ \mu(E_n) < \infty \implies \mu(E) = \lim_{n\to\infty} \mu(E_n)$$

Outer Measures

Definition: *Given a set X, function $\mu^* : \mathcal{P} \to [0, \infty]$ is called an outer measure if*

- $\mu(\emptyset) = 0$
- *Monotonicity*

$$E \subseteq F \implies \mu^*(E) \leq \mu^*(F)$$

- *If $\{E_n\}_{n\in\mathbb{N}}$ are sets in S then*

$$\mu^*\left(\bigcup_{n\in\mathbb{N}} E_n\right) \leq \sum_{n\in\mathbb{N}} \mu^*(E_n)$$

Given an arbitrary outer measure we get the sigma algebra of Caratheodory measurable sets given by

$$C_X = \{E \in \mathcal{P}(X) : \forall A \in \mathcal{P}(X)\ \ \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E)\}$$

Note that this is precisely when the disjoint sets $A \cap E$ and $A \setminus E$ are additive.

Theorem. *Given a set X and an outer measure μ^* then the set C_X of Caratheodory measureable sets is a σ-algebra moreover $\mu = \mu^*|_{C_X}$ is a complete measure on this σ-algebra*

Proof. It is clear that $X, \emptyset \in C_X$.

Step 0: Compliments

Let $E \in C_X$ then

$$\begin{aligned}\mu^*(A) &= \mu^*(A \cap E) + \mu^*(A \setminus E)\\ &= \mu^*(A \cap E) + \mu^*(A \cap E^C)\\ &= \mu^*(A \cap E^C) + \mu^*(A \setminus (E^C)^C)\\ &= \mu^*(A \cap E^C) + \mu^*(A \setminus E)\end{aligned}$$

Step 1: Finite Unions:

Let $E, F \in C_X$ and $A \subseteq X$ arbitrary. Then

$$\begin{aligned}\mu^*(A) &= \mu^*(F \cap A) + \mu^*(F^C \cap A)\\ &= \mu^*(E \cap F \cap A) + \mu^*(E^C \cap F \cap A) + \mu^*(E \cap F^C \cap A) + \mu^*(E^C \cap F^C \cap A) \quad \text{(measurability of E)}\\ &\geq \mu^*(((E \cap F) \cup (E \cap F^C) \cup (E^C \cap F)) \cap A) + \mu^*((E \cup F)^C \cap A) \quad \text{(Demorgans law, subadditivity)}\\ &= \mu^*((E \cup F) \cap A) + \mu^*((E \cup F)^C \cap A)\end{aligned}$$

and

$$\begin{aligned}\mu^*(A) &= \mu^*(A \cap (E \cup F) \cup A \cap (E \cup F)^C)\\ &\leq \mu^*(A \cap (E \cup F)) + \mu^*(A \cap (E \cup F)^C) \quad \text{(subadditivity)}\end{aligned}$$

hence

$$\mu^*(A) = \mu^*(A \cap (E \cup F)) + \mu^*(A \cap (E \cup F)^C)$$

and the union is therefore measurable.

Step 2: Countable Unions of Disjoint Sets:

Let $\{E_k\}$ be a countable collection of pairwise disjoint Caratheodory measurable sets. Then let

$$G_n = \bigcup_{i=1}^{n} E_i$$

and we have that

$$G_n \nearrow G = \bigcup_{i \in \mathbb{N}} E_i$$

And each G_n is measurable by step 1.

Now because E_n is measurable we have

$$\begin{aligned}\mu^*(G_n \cap A) &= \mu^*(E_n \cap G_n \cap A) + \mu^*(E_n^C \cap G_n \cap A)\\ &= \mu^*(E_n \cap A) + \mu^*(G_{n-1} \cap A)\\ &= \sum_{k=1}^{n} \mu^*(E_k \cap A)\end{aligned}$$

Where the second equality is from disjointness and the third is from induction.

Now $G^C \subseteq G_n^C$ (for any n) hence by subadditivity of outer-measures

$$\sum_{k=1}^{n} \mu^*(E_k \cap A) + \mu^*(G^C \cap A) \leq \mu^*(A)$$

This is for any n and hence holds in the limit

$$\mu^*(G \cap A) + \mu^*(G^C \cap A) \leq \sum_{k=1}^{\infty} \mu^*(E_k \cap A) + \mu^*(G^C \cap A) \leq \mu^*(A)$$

And just as in the finite case the reverse inequality is immediate hence G is measurable.

Step 3: Countable Unions:

Follows immediately from step 2 by disjointifying the sequence.

Showing μ is a complete measure:

From the axioms of an outer measure all we need to show is that the restriction of μ^* to Caratheodory measurable sets is additive on disjoint sets. So let $\{E_k\}$ be a sequence of pairwise disjoint measurable sets. Using the calculation from finite unions and setting A to be $E_i \cup E_j$ we get

$$\mu(E_i \cup E_j) = \mu(E_i \cap (E_i \cup E_j)) + \mu(E_i^C \cap (E_i \cup E_j)) = \mu(E_i) + \mu(E_j)$$

And the calculuation from pairwise disjoint sets setting $A = G = \cup E_k$ gives

$$\sum_{k=1}^{n} \mu^*(E_k \cap G) + \mu^*(G^C \cap G) \leq \mu^*(G)$$

$$\implies \sum_{k=1}^{n} \mu(E_k) \leq \mu(\cup E_k) \leq \sum_{k=1}^{n} \mu(E_k)$$

For any n hence in the limit. So μ is in fact a measure.

Finally we show completeness. Assume E is measurable with $\mu(E) = 0$ and $F \subseteq E$. Then for any A

$$\begin{aligned} F &\subseteq E \\ F \cap A &\subseteq E \cap A \subseteq E \\ \implies \mu^*(F \cap A) &\leq \mu^*(E \cap A) \leq \mu^*(E) = 0 \end{aligned}$$

Moreover $A \cap B \subseteq A$ so we get

$$\mu^*(A \cap F^C) \leq \mu^*(A) = \mu^*((A \cap F^C) \cup (A \cap F)) \leq \mu^*(A \cap F^C) + \mu^*(A \cap F) = \mu^*(A \cap F^C)$$

Hence F is measurable and moreover it has measure zero (set $A = F$)

Metric Outer Measures

On a metric space the topology is generated by the balls. There is then a unique smallest sigma algebra generated on this topology, which we call the Borel sigma algebra on the metric space. A measure on a metric space with the Borel sigma algebra is called a Borel measure.

Recall that in a metric (X, d) space there is a natural way to extend the metric to subsets of X by

$$d(A, B) = inf\{d(x, y) : x \in A, y \in B\}$$

Definition: *An outer measure is a metric outermeasrue iff whenever $d(A, B) > 0$ we have*

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$$

In particular the Lebesgue outer measure is a metric outer measure.

Theorem. *Given a metric outer measure, μ^* on (X, d), the Borel sets are Caratheodory measurable.*

Proof. The Borel sets are generated by closed (or open sets) hence it will suffice to show that all closed subsets of X are measurable. Moreover because the reverse inequality is immediate from subadditivity it suffices to show that

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^C)$$

We can assume WLOG that $\mu^*(A) < \infty$ (otherwise it is trivial).

Let A be given (with finite outer measure) and let E be closed, then set $A_n = \{x \in A : d(x, E) \geq \frac{1}{n}\}$. Note that the sequence $\{A_n\}$ is increasing ($A_n \subseteq A_{n+1}$) and $A \setminus E = \cup A_n$; this follows because E is closed (when E is does not contain all its limit points then the left hand side will contain the limit points while the right hand side will not). Then

$$d(A \cap E, A_n) \geq \frac{1}{n}$$
$$\implies \mu^*(A) \geq \mu^*((A \cap E) \cup A_n) = \mu^*(A \cap E) + \mu^*(A_n)$$

Using the fact that μ^* is a metric outer measure and $(A \cap E) \cup A_n \subseteq A$ by definition.

So if we can show that $\mu^*(A_n)$ approaches $\mu^*(A \cap E^C)$ we are done.

Set $B_n = A_{n+1} \setminus A_n$ (small anuli appraching the boundary of E), notice that if $x \in A_n$ and $y \in B_{n+1}$ we have $d(x, E) \geq 1/n$ and $d(y, E) \geq 1/(n + 1)$. Hence

$$\frac{1}{n} \leq d(x, E) \leq d(x, y) + d(y, E) \leq d(x, y) + \frac{1}{n + 1}$$

Thus because x and y are arbitrary

$$d(B_{n+1}, A_n) \geq \frac{1}{n} - \frac{1}{n + 1}$$

μ^* is a metric outer measure so we get

$$\mu^*(A_{2k+1}) \geq \mu^*(B_{2k} \cup A_{sk-1}) = \mu^*(B_{2k}) + \mu^*(A_{2k-1})$$

Hence by a similar induciton to earlier

$$\mu^*(A_{2k+1}) \geq \sum_{j=1}^{k} \mu^*(B_{2j})$$

Similarly

$$\mu^*(A_{2k}) \geq \sum_{j=1}^{k} \mu^*(B_{2j-1})$$

By the finiteness of the measure of A both of these sums converge. Now apply monotonicity and subadditivity gives

$$\mu^*(A_n) \leq \mu^*(A \setminus E) \leq \mu^*(A_n) + \sum_{k=n+1}^{\infty} \mu^*(B_k)$$

And because the sum is convergent the tails must go to zero so we get that

$$\mu^*(A_n) \leq \mu^*(A \setminus E) \leq \mu^*(A_n)$$

And we are done.

Theorem. *Given a metric space X with a Borel measure* μ *such that for any* $x \in X, r \in \mathbb{R}^+$ *we have* $\mu(B_r(x)) < \infty$. *Then for any Borel set E and any* $\epsilon > 0$

- *there is an open set U such that* $E \subseteq U$ *and* $\mu(U \setminus E) < \epsilon$
- *There is a closed set F such that* $F \subseteq E$ *and* $\mu(E \setminus F) < \epsilon$

proof

Measurable Functions

Definition: *A function between two measurable spaces $f : (X, A) \to (Y, B)$ is measurable iff for every $b \in B$ $f^{-1}(b) \in A$. Or the preimage of measurable sets are measurable.*

Brian definies measurable functions into $\mathbb{R}$ or the extended reals by implicitly giving them the Borel sigma algebra. In particular if $f, g : (X, A) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ are measurable then

- If in addition f and g are finite valued then $f^k, f + g, \alpha f, fg, f/g$ are measurable
- If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of measurable functions then sup, inf, limsup and liminf are all measurable functions. If the pointwise limit exists it is measurable
- If (X, A, μ) is complete and $h : X \to \mathbb{R}$ agrees with f a.e. (there is a measure zero set, such that f=h on its compliment in X) then h is measurable.
- If $A = \mathcal{B}_X$ (Borel sigma algebra) then every continuous function is measurable

Approximating Measurable Functions

Fix a measure space (X, A, μ)

Definition: *A function $\phi : X \to \mathbb{R}$ is simple iff it is measurable and there is some $a_1, ..., a_n \in \mathbb{R}$ and $E_1, ..., E_n \in A$ such that*

$$\phi = \sum_{k=1}^{n} a_k \chi_{E_k}$$

Theorem. *If $f : X \to [-\infty, \infty]$ is measurable then*

1. *If f is non-negative there is an increasing sequence of non-negative simple functions on X $\{\phi_k\}_{k\in\mathbb{N}}$ that converge pointwise to f i.e.*

$$\text{For any } x\ \phi_k(x) \leq \phi_{k+1}(x) \text{ and } \lim_{k\to\infty} \phi_k(x) = f(x)$$

2. *There is a sequence of simple functions such that $|\phi_k(x)| \leq |\phi_{k+1}(x)|$ converging pointwise to f.*
3. *If (X, A, μ) is sigma finite then the above is still true if we add the extra condition that the ϕ_k are supported on sets of finite measure.*

Proof.

Claim 1:

Let $F_N : X \to \mathbb{R}$ be $F_N(x) = min(f(x), N)$. It is clear that $F_N \to f$ as $n \to \infty$. Now let

$$E_{\ell,M} = \{x \in X : \frac{\ell}{M} < F_N(x) \leq \frac{\ell+1}{M}\}$$

which makes the following a simple function

$$F_{N,M}(x) = \sum_{t=0}^{NM-1} \frac{\ell}{M} \chi_{E_{\ell,M}}(x)$$

Then for any $k \in \mathbb{N}$ we can set $N = M = 2^k$ and define $\phi_k = F_{2^k,2^k}$.

This is similar to assignment 2.

Claim 2:

Recall $f^+(x) = max(f(x), 0)$ and $f^-(x) = max(-f(x), 0)$, and that $f = f^+ - f^-$. Then notice that they are both positive and apply part 1 to get two increasing sequences $\{\phi_k^+\} \to f^+$ and $\{\phi_k^-\} \to f^-$ finally verify that

$$\phi_k = \phi_k^+ - \phi_k^-$$

Claim 3:

We just change the construction in (1) to $F'_N(x) = \chi_{X_N} F_N$ where $X = \cup X_N$ and $\mu(X_N) < \infty$ for every N (σ-finite).

Theorem (Egorov)**.** *If E is a measurable set with finite measure and $\{f_k\}_{k\in\mathbb{N}}$ is a sequence of measureable functions $f_k : E \to \mathbb{R}$ converging pointwise almost everywhere to f on E, then for any $\epsilon > 0$ there is a measurable set $A \subseteq E$ such that $m(E \setminus A) < \epsilon$ and $f_k \to f$ uniformely on A*

Proof. Fix an ϵ and an $n \in \mathbb{N}$. Pointwise convergence tells us that for a.e. x there is a $k(x) \in \mathbb{N}$ such that for every $j > k(x)$ we have

$$|f_j(x) - f(x)| < \frac{1}{n}$$

Define

$$E_k^n = \{x \in E : |f_j(x) - f(x)| < \frac{1}{n} \ \ \forall j > k\}$$

Clearly this forms an increasing sequence in k, and by pointwise convergence a.e. in E we have that

$$E = N \cup \bigcup_{k\in\mathbb{N}} E_k^n$$

where $m(N) = m(\{\text{ points of non-convergence}\}) = 0$. i.e. $E_k^n \cup N \nearrow E$ hence $m(E) = \lim_{k\to\infty} m(E_k^n \cup N) = \lim_{k\to\infty} m(E_k^n)$ (by disjointness of N and all E_k^n). Because $m(E) < \infty$ we know

$$\lim_{k\to\infty} m(E \setminus E_k^n) = \lim_{k\to\infty} (m(E) - m(E_k^n)) = 0$$

Hence for every $n \in \mathbb{N}$ we get a k_n such that

$$m(E \setminus E_{k_n}^n) < \frac{1}{2^n}$$

Now set

$$\tilde{A} = \bigcap_{n=M}^{\infty} E_{k_n}^n$$

By countable subadditivity and taking M sufficiently large

$$m(E \setminus \tilde{A}) \leq \sum_{n=M}^{\infty} m(E \setminus E_{k_n}^n) \leq \sum_{n=M}^{\infty} 2^{-n} = 2^{1-M} < \epsilon$$

It is clear that $\tilde{A}$ is measurable (intersection of measurable sets) so if we can show uniform convergence on it we will be done:

Let $\delta > 0$ be given and select $n \in \mathbb{N}$ such that $\frac{1}{n} < \delta$. If $x \in \tilde{A}$ we have for every $j > k_n$

$$|f_j(x) - f(x)| < \frac{1}{n} < \delta$$

Because either $n < M$ in which case

$$|f_j(x) - f(x)| < \frac{1}{M} < \frac{1}{n} < \delta$$

or $n > M$ in which case x is already in the set satisfying this condition just by definition of $\tilde{A}$ intersecting all those sets.

Integration

For simplicity assume that (X, A, μ) is a sigma finite measure space. Denote the integral of a measurable function $f : X \to \mathbb{R}$

$$\int_X f(x) d\mu(x)$$

When it is not ambigous which variable, domain or measure we are integrating then those symbols will be dropped.

To define integration we follow a four step program

1. Simple functions with finite measure support (using sigma finite)
2. Bounded emasurable function with finite support
3. Non-negative measurable functions
4. Integrable functions

Simple Functions

Take some simple function on X

$$\phi = \sum_{k=1}^{n} a_k \chi_{E_k}$$

then define

$$\int_X \phi d\mu = \sum_{k=1}^{n} a_k \mu(E_k)$$

One checks that this is well definied and independent of representation for the simple function.

Bounded Measurable Functions

Let $f : X \to \mathbb{R}$ be a bounded function with a finite measure support. Then we can approximate it by a sequence of simple functions $\{\phi_k\}_{k \in \mathbb{N}}$ that converge to f pointwise a.e. Then define

$$\int_X f d\mu = \lim_{n \to \infty} \int_X \phi_n d\mu$$

Using Ergorovs theorem we check this is well definied and independent of choice of sequence.

Non-negative Measurable Functions

$$\int_X f d\mu = sup \left\{ \int_X g d\mu : g : X \to [0, \infty], \text{ g is supported on a set of finite measure and } 0 \le g \le f \right\}$$

Integrable Functions

Let $f : X \to \mathbb{R}$ measurable, then f is integrable iff $\int |f| < \infty$. We write $f = f^+ - f^-$ where $f^+ = max\{0, f\}$ and $f^- = max\{0, -f\}$. Then for integrable f define

$$\int f = \int f^+ - \int f^-$$

well definedness of all the integral definitions

Integration Theorems

additivity, monotone, triangle inequality for integrals

Integrals are linear, additive, monotone and satisfy the triangle inequality.

Theorem (Bounded Convergence). *Let $M \in \mathbb{R}^+$ a constant and E be a set of finite measure. If $\{f_n : X \to \mathbb{R}\}$ is a sequence of measurable functions that are uniformely bounded by M for all $x \in X$, have support contained in E and converge p.w. a.e. to f then f is bounded and supported up to a set of measure zero on E. Moreover*

$$\lim_{n\to\infty} \int |f_n - f| = 0$$

hence we can interchange limits.

Proof.

$$\forall x, n \;\; f_n(x) \le M \implies \lim_{n\to\infty} f_n(x) = f(x) \le M$$

Moreover the support of f must agree with the support of f_n for all n a.e. It remains only to show that the integral is zero.

Given an $\epsilon > 0$ we apply Egorovs theorem to obtain a measurable set $A \subseteq E$ such that $m(E \setminus A) < \epsilon$ and $f_n \to f$ uniformely on A. i.e. there is an $N \in \mathbb{N}$ such that $\forall x \in X, \forall n > N$

$$|f_n(x) - f(x)| < \epsilon$$

Hence for $n > N$ we have

$$\begin{aligned}\int |f_n - f| &\le \int_A |f_n - f| + \int_{E\setminus A} |f_n - f| \\ &\le \epsilon m(A) + 2Mm(E \setminus A) \\ &\le \epsilon m(E) + 2M\epsilon \to 0\end{aligned}$$

Where it is important for these calculations to make sense that $m(E) < \infty$

Lemma (Fatou). *$\{f_n\}$ a sequence of non-negative measurable functions on X converging p.w.a.e to f then*

$$\int_X f d\mu \le \liminf_{n\to\infty} \int_X f_n d\mu$$

Proof. We take a function $g : X \to \mathbb{R}$ that is bounded, measurable, supported on a set of finite measure $\mathcal{S}(g)$ and $0 \le g \le f$. Now let $g_n(x) = min(f_n(x), g(x))$. Notice that these are also bounded, measurable and supported on $\mathcal{S}(g)$ for every n. Moreover $g_n \to g$ p.w. so by the Bounded convergence theorem

$$\int g = \lim_{n\to\infty} \int g_n$$

By monotonicity of integration we also have for every n

$$\int g_n \le \int f_n$$

$$\implies \int g \le \liminf_{n\to\infty} \int f_n$$

$$\implies \sup\left\{\int g : \text{ g is bounded measurable finite measure support } 0 \le g \le f\right\} = \int f \le \liminf_{n\to\infty} \int f_n$$

Notice that the lim inf is doing something because the limit may not exists, where the lim inf will.

Theorem (MCT). *$\{f_n\}$ a sequence of non-negative measurable functions on X such that $f_n(x) \le f(x)$ and $f_n \to f$ p.w.a.e then*

$$\int_X f d\mu = \lim_{n\to\infty} \int_X f_n d\mu$$

Proof. Monotonicity of integrals tells us immediately that

$$\limsup_{n\to\infty} \int f_n \leq f$$

Now applying Fatous lemma we get

$$\int f \leq \liminf_{n\to\infty} \int f_n$$

strictly the MCT is if $f_n \nearrow f$ but this is clearly more general.

Theorem (Dominated Convergence)**.** *$\{f_n\}$ a sequence of measuable functions converging p.w.a.e. to f on X. If there is some integrable function g such that $|f_n(x)| \leq g(x)$ a.e. and all $n \in \mathbb{N}$ then*

$$\int_X f d\mu = \lim_{n\to\infty} \int_X f_n d\mu$$

Proof. Because $|f_n(x)| \leq g(x)$ we know that $|f(x)| \leq g(x)$ a.e. moreover by monotonicity of the integral both f and f_n are integrable, hence finite a.e.

Now $|f_n| \leq g \implies f_n \leq g \implies g - f_n \geq 0$ a.e., likewise for $g - f \geq 0$ a.e. moreover

$$g - f_n \to g - f$$

p.w.a.e. thus applying Fatou

$$\int (g - f) \leq \liminf_{n\to\infty} \int (g - f_n) = \int g - \limsup_{n\to\infty} \int f_n$$

(where passing the inf through the negative makes it a sup), rearranging gives

$$\limsup_{n\to\infty} \int f_n \leq \int f$$

Applying the same trick we get

$$g + f_n \to g + f$$

where both are non-negative functions hence

$$\int f \leq \liminf_{n\to\infty} \int f_n$$

Lp Spaces

We define an equivilence relation on measuable functions, by declaring $f \sim g$ iff $f(x) = g(x)$ a.e. Then we define

$$L^p(X, A, \mu) = \{[f]_\sim : \text{ f is measurable and } \int_X |f|^p d\mu < \infty\}$$

$L^p(X, A, \mu)$ is a Banach space when given the norm

$$\|f\|_{L^p} = \left(\int_X |f|^p d\mu\right)^{\frac{1}{p}}$$

For $p = \infty$ we define $L^\infty(X, A, \mu)$ the space of uniformely bounded a.e. functions with the norm being the inf of all uniform bounds.

Theorem (Riesz-Fischer)**.** *$L^p(\Omega)$ is complete.*

proof of completeness of Lp

Simple functions, step functions, continuous functions of compact support are all dense in L^p. By taking step functions over rational coefficients and rational sets we get a countable dense subset (seperable).

Proof of the density of certain functions in Lp. Seperable for general Lp?

Inequalities

All norms here are L^p for $p \in [1, \infty]$

Lemma (Youngs Inequality). *Given $p \in (1, \infty)$ and its conjugation q, then for any $a, b \in \mathbb{R}^+$*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

with equality iff $a^p = b^q$

Proof. Let $g : [1, \infty) \to \mathbb{R}$ be given by

$$g(x) = \frac{1}{p}x^p + \frac{1}{q} - x$$

(notice that this is the equation for b=1), then

$$g'(x) = x^{p-1} - 1 \geq 0$$

because $x \geq 1$. Notice that $g(1) = 0$ because p and q are conjugate so

$$g(x) \geq 0$$

or equivilently

$$x \leq \frac{1}{p}x^p + \frac{1}{q}$$

for $x \geq 1$. Notice that equality attains for $x = 1$. Now let $x = ab^{1-q}$ and multiply both sides by b^q to obtain the inequality. And notice that equality is only for $ab^{1-q} = 1 \iff a^p = b^{p(q-1)} = b^q$ (using conjugacy).

Theorem (Holders). *Let p and q be conjugate then for $f \in L^p, g \in L^q$ then*

$$||fg||_{L^1} \leq ||f||_{L^p}||g||_{L^q}$$

Proof.

p = 1: Then $q = \infty$ and

$$|f(x)g(x)| \leq |f(x)|||g||_{L^\infty}$$

for a.e. x hence the inequality holds by monotonicity and lineararity of the intergral.

$\mathbf{1 < p < \infty}$**:** It will suffice to prove that if $||f||_p = ||g||_q = 1$ then $||fg||_1 \leq 1$. This is because for arbitrary f and g we can normalise them by dividing by their norms (WLOG they are non-zero becuase the inequality is trivial if they were).

Apply youngs inquality to $a = |f(x)|, b = |g(x)|$ to get

$$|f(x)g(x)| \leq \frac{1}{p}|f(x)|^p + \frac{1}{q}|g(x)|^q$$

Then integrate to get

$$\int |f(x)g(x)| \leq \frac{1}{p}\int |f(x)|^p + \frac{1}{q}\int |g(x)|^q = \frac{1}{p} + \frac{1}{q} = 1$$

We can also trace the equality condition from Youngs inequality through the argument to arrive at the fact that equality attains iff $c|g(x)|^q = |f(x)|^p$ (a.e. x and some c constant).

For an $f \in L^p \setminus \{0\}$ we define $f^*(x) = \frac{|f(x)|^{p-2}f(x)}{||f||_{L^p}^{p-1}}$.

Lemma. *f^* is the unique L^q function such that*

$$\int ff^* = \|f\|_{L^p} \text{ and } \|f^*\|_{L^q} = 1$$

Proof. The two properties are by direct computation.
For uniqueness: Let g be another function satidfying the hypotheses. Then

$$\|f\|_p = \int fg \leq \int |fg| \leq \|f\|_p \|g\|_q = \|f\|_p$$
$$\implies \|f\|_p = \int fg = \int |fg| = \|f\|_p \|g\|_q$$

So in particular using our equivilent conditions for equality from Holders above we know that $fg \geq 0$ a.e. and $|g(x)|^q = c^{\frac{1}{q}}|f(x)|^{p-1}$ for some c a.e.. Hence $g = \alpha |f|^{p-2} f$ for some $\alpha \in \mathbb{R}^+$ (because g and f have the same sign by the first equality, we can peel one of the powers of the absolute value to recover g), which along with the condition of integrating to one fixes g to be f^*.

Theorem (Minkowski Inequality)**.**

$$\|f + g\| \leq \|f\| + \|g\|$$

Proof.

p=1: Immediate using the triangle inequality and linearity of the integral.

$1 < p < \infty$: WLOG we assume $f + g \neq 0$ then

$$\begin{aligned} \|f + g\|_p &= \int (f+g)(f+g)^* \\ &= \int f(f+g)^* + \int g(f+g)^* \\ &\leq \|f\|_p \|(f+g)^*\|_q + \|g\|_p \|(f+g)^*\|_q \qquad (Holder) \\ &= \|f\|_p + \|g\|_p \end{aligned}$$

Extension Theorem

Definition: *A ring R over a set X is a collection of subsets of X such that*

- $\emptyset \in R$
- $E, F \in R \implies E \setminus F \in R$
- *Closed under finite unions*
- *Closed under finite intersections*

An algebra of sets is a Ring that contains X.

Definition: *Given a set X and a ring of subsets R then a function $\mu_0 : R \to [0, \infty]$ is a premeasure iff*

- $\mu_0(\emptyset) = 0$
- *Countable additivity of pairwise disjoint sets*

Note that this is not a measure yet because R is not a sigma algebra. Also note that it is monotone (follows from additivity).

Theorem. *Given (X, R, μ_0) a pre-measure space we can define $\mu^* : \mathcal{P}(X) \to [0, \infty]$ as*

$$\mu^*(E) = \inf \left\{ \sum_{n \in \mathbb{N}} \mu_0(E_j) : E \subset \bigcup_{n \in \mathbb{N}} E_n \text{ and } E_j \in R \right\}$$

- *μ^* is an outer measure*
- $\mu^*|_R = \mu_0$
- *All sets in R are caratheodory measurable (wrt μ^*)*

Proof.

Outer Measure: Empty set is clear (its in the ring), monotonicity is clear (covers cover). So we check countable subadditivity.

Let $\{E_n\}$ be a collection of sets in X. We assume WLOG (it is immediate otherwise) that there is a countable cover of sets in the ring for each E_n. Then for any $\epsilon > 0$ we can take a cover of E_n call it E_n^j such that

$$\sum_j \mu_0(E_n^j) \le \mu^*(E_n) + \epsilon/2^n$$

Then summing over n gives

$$\mu^*(\cup_n E_n) \le \sum_{n,j} \mu_0(E_n^j) \le \sum_n \mu^*(E_n) + \epsilon \to \sum_n \mu^*(E_n)$$

Agreement on the Ring: Let $E \in \mathcal{R}$ be arbitrary. It is immediate that $\mu^*(E) \le \mu_0(E)$ (it covers itself) so we have to show the reverse inequality. WLOG we take a pairwise disjoint collection of sets in $\mathcal{R}$, call them $\{F_k\}$, that cover E precisely (a cover exists, because other wise the inequality is trivial, ring axioms allow us to disjointify the collection and we can just intersect with E). By monotonicity of μ_0:

$$\mu_0(E) = \sum_k \mu_0(F_k) \le \sum_k \mu_0(F_k)$$

Hence the inf over such sums gives the result.

Caratheodory Measurable: Let $E \in \mathcal{R}$ and $A \subseteq X$ be arbitrary. WLOG $\mu^*(A) < \infty$. So take a cover by sets in the ring $\{A_j\}$ and notice that $\{A_j \cap E\}, \{A_j \cap E^C\}$ are covers of $A \cap E, A \cap E^C$ respectively. Then

$$\begin{aligned}\sum_j \mu_0(A_j) &= \sum_j [\mu_0(A_j \cap E) + \mu_0(A_j \cap E^C)] \\ &= \sum_j \mu_0(A_j \cap E) + \sum_j \mu_0(A_j \cap E^C) \\ &\geq \mu^*(A \cap E) + \mu^*(A \cap E^C)\end{aligned}$$

Hence by taking inf over all such covers

$$inf\{\sum_j \mu_0(A_j)\} = \mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^C)$$

As usual the oposite inequality follows from the subadditivity of the outermeasure and therefore we have equality.

Note by convention $inf(\emptyset) = \infty$.

Theorem (Caratheodory Extension Theorem)**.** *Let (X, R, μ_0) a pre-measure space. There is a smallest sigma algebra containing R, call it A. Then*

- *There exists a measure $\mu : A \to [0, \infty]$ that restricts to μ_0 on R.*
- *If further there is a countable cover of X by sets of finite μ_0 pre-measure then this measure μ is unique.*

Proof. We have the outer-measure above that we then restrict to Caratheodory measurable sets.

Uniqueness: Let ν be another measure on A such that $\forall E \in \mathcal{R}$ $\nu(E) = \mu_0(E)$. Consider $F \in A$ WLOG with finite $\mu-$measure (this is the point where we are using the μ_0 sigma finitness because if F is infitire then we approximate it by intersecting with finite measure sets that union to be the whole space and use continuitiy of measure to get the measure in the limit). Take a collection of ring elements that covers F, $\{E_j\}$ then

$$\nu(F) \leq \sum_j \nu(E_j) = \sum_j \mu_0(E_j)$$

Hence $\nu(F) \leq \mu(F)$. Now take $\epsilon > 0$ and let $\{E_j\}$ be a cover of F by ring sets such that

$$\sum_j \nu(E_j) \leq \mu(F) + \epsilon$$

Let $E = \cup_j E_j$ then by the finite meaesure of F we have that $\mu(E \setminus F) \leq \epsilon$. Then

$$\mu(E) = \lim \mu(\cup_{j=1}^n E_j) = \lim \nu(\cup_{j=1}^n E_j) = \nu(E)$$

Hence

$$\mu(F) \leq \mu(E) = \nu(E) = \nu(F) + \nu(E \setminus F) \leq \nu(F) + \mu(E \setminus F) \leq \nu(F) + \epsilon \to \nu(F)$$

Product Measures

Given an algebra R we denote

$$R_\sigma = \{ \text{ countable unions of sets in R } \}$$

$$R_{\sigma\delta} = \{ \text{ countable intersections of sets in } R_\sigma \}$$

Lemma. *Let (X, R, μ_0) a pre-measure space where $X \in R$ (R is an algebra). Then let μ^* be the induced outermeasure.*
For every E and every $\epsilon > 0$ there is an $S \in R_\sigma$ such that $E \subseteq S$ and $\mu^*(S) \leq \mu^*(E) + \epsilon$
There is also a $T \in R_{\sigma\delta}$ such that $E \subseteq T$ and $\mu^*(T) = \mu^*(E)$

proof

Let (X_1, A_1, μ_1) and (X_2, A_2, μ_2) be complete sigma finite measure spaces. Consider $R = \left\{\bigcup_{k=1}^n B_k \times B_k' : B_k \in A_1, B_k' \in A_2\right\}$, the collection of finite unions of "rectangles" from the two measure space.

Lemma. *R is an algebra*

proof

Lemma. *The following definies a premeasure on R,*

$$\mu_0(\bigcup_{k=1}^n B_k \times B_k') = \sum_{k=1}^n \mu_0(B_k \times B_k')$$

with base case given by

$$\mu_0(B \times B') = \mu_1(B)\mu_2(B')$$

proof

Now we apply the two extension theorems to get an outer measure and then a unique measure given by the restriction of this outer measure. This gives a measure $\mu : \sigma(R) \to [0, \infty]$ (on the sigma algebra generated by R).

If we drop assumptions of sigma finiteness then this still goes through however we lose uniqueness. Some theorems about product measures will also fail.

Theorems About Product Measures

Lemma. *If E is measurable in $X_1 \times X_2$ then*

- $E^y = \{x \in X_1 : (x, y) \in E\}$ *is measurable in X_1 for a.e. $y \in X_2$*
- $y \mapsto \mu_1(E^y)$ *is a measurable function on X_2*
- $\int_{X_2} \mu_1(E^y) d\mu_2(y) = (\mu_1 \times \mu_2)(E)$

proof

Theorem (Fubinis)**.** *If $f : X_1 \times X_2 \to \mathbb{R}$ is $\mu_1 \times \mu_2$ measurable then*

- $f^y : X_1 \to \mathbb{R}, x \mapsto f(x, y)$ *is integrable on X_1 a.e. $y \in X_2$*
- $y \mapsto \int_{X_1} f^y d\mu_1$ *is an X_2 integrable function*
- $$\int_{X_2} \int_{X_1} f^y d\mu_1(x) d\mu_2(y) = \int_{x_1 \times X_2} f(x, y) d(\mu_1 \times \mu_2)(x, y)$$

proof

Decompositions

Signed Measures

Definition: *A signed measure on a measurable space (X, A) is a function $\nu : A \to [-\infty, \infty]$ sotisfying*

- *ν takes at most one of the values $\pm\infty$*
- $\nu(\emptyset) = 0$
- *Countable additivity: Let $\{E_n\}$ be a countable collection of pairwise disjoint sets in A, then we have that*

$$\nu(\cup E_n) < \infty \implies \sum |\nu(E_n)| < \infty$$

and

$$\nu(\cup E_n) = \sum |\nu(E_n)|$$

Given a signed measure ν on a measurable space (X, A) then a measurable set E is

- positive if for every measurable set $F \subseteq E$ we have $\nu(F) \geq 0$
- negative if for every measurable set $F \subseteq E$ we have $\nu(F) \leq 0$
- null if for every measurable set $F \subseteq E$ we have $\nu(F) = 0$

The intersection of a positive and negative set is a null set.

Lemma. *A countable union of positive sets is positive*

Proof. Let $\{E_k\}$ be a countable collection of positive sets, whose union is E. Then let $F \subseteq E$ arbitrary. We need to show it has positive $\nu-$measure.

Let $F_1 = E_1 \cap F$ and $F_i = F \cap E_i \setminus \cup_{j=1}^{n-1} F_j$, these are measurable, pairwise disjoint and $F_i \subseteq E_i$. But E_i is positive hence F_i is positive. Thus by countable additivity:

$$\nu(F) = \nu(\bigcup_i F_i) = \sum_i \nu(F_i) \geq 0$$

Hahn Decomposition

Lemma (Hahns Lemma)**.** *Given a signed measure space (X, A, ν), if a set has positive (finite) measure then there is a positive set of nonzero measure contained in it i.e.*

$$E \text{ measurable and } 0 < \nu(E) < \infty \implies \exists A \subseteq E \text{ a positive set such that } \nu(A) > 0$$

Proof. If E is positive we are done. So assume E is not positive, hence there is some $F \subseteq E$ such that $\nu(F) < 0$. Let $m_1 \in \mathbb{N}$ be the smallest positive integer such that

$$\exists F \subseteq E \quad \nu(F) \leq \frac{-1}{m_1}$$

Such an integer exists by hypothesis. Notice that from the definition

$$\forall F \subseteq E \quad \nu(F) > \frac{-1}{m_1 - 1}$$

where the right is $-\infty$ for $m_1 = 1$. So let E_1 be such that

$$E_1 \subseteq E \quad and \quad \nu(E_1) \leq \frac{-1}{m_1}$$

Then inductively define m_n and E_n as the smallest positive integer such that there is some set disjoint from the previous E_i that has measure smaller that $-1/m_n$ and E_n is one such set.

Case 1: Terminates after finitely many sets: Then

$$A = E \setminus \bigcup_{i=1}^{n} E_k$$

is positive (because by definition all subsets must have positive measure), moreover

$$\nu(A) = \nu(E) - \sum \nu(E_k) > 0$$

Becasue E is positive and each of the $\nu(E_k) < 0 \implies -\nu(E_k) > 0$. (this used the finiteness of the measure of E).

Case 2: Does not terminate: We get an infinite sequence of measurable stes $\{E_k\}$ and again define

$$A = E \setminus \cup E_k$$

Again the measure of A is positive (as above) so it remains to show that all subsets have positive measure. So let $F \subseteq A$ measurable. Now we know that subsets of finite measure (signed) sets have finite measure hence $|\nu|(\cup E_k) < \infty$ so

$$-\infty < \nu(\cup E_k) \leq \sum \nu(E_k) \leq -\sum \frac{1}{m_n}$$

hence $\sum \frac{1}{m_n} < \infty$ so the tails go to zero so

$$F \subseteq A \subseteq E \setminus \bigcup_{k=1}^{n-1} E_k$$

which by definition of m_n tells us $\nu(F) > \frac{-1}{m_n - 1} \to 0$ in the limit. So $\nu(F) > 0$ and A is positive.

Theorem (Hahn Decomposition Theorem). *There is a positive set P and a negative set N wrt ν such that*

$$X = P \cup N \;\;\&\;\; P \cap N = \emptyset$$

Proof. WLOG we assume that $\nu < \infty$ (ν can only take one of $\pm\infty$ and I assume the other case is similar).

There is a positive set with maximum measure: Let $\lambda = \sup\{\nu(E) : E \text{ is a positive set }\}$. Notice that $\nu(\emptyset) = 0$ so there is at least one positive set. There are positive sets $\{P_n\}_{n\in\mathbb{N}}$ such that $\nu(P_n) \to \lambda$, now define $P = \cup P_n$. By a lemma above P is positive and hence $P \setminus P_n$ is positive (its a subset).

$$\nu(P) = \lim_{n\to\infty} \nu(P_n \cup P \setminus P_n) \geq \lim_{n\to\infty} \nu(P_n) = \lambda$$

Hence P has maximal ν measure. By our finiteness assumption also $\lambda < \infty$

X\P Is Negative: Let $N = X \setminus P$ and suppose that N is not negative. i.e. there is some $E \subseteq N$ such that $\nu(E) > 0$, then by Hahns Lemmathere is a positive set $A \subseteq E \subseteq N$ such that $\nu(A) > 0$ but then $P \cup A$ is positive and

$$\nu(P \cup A) = \nu(P) + \nu(A) > \lambda$$

A contradiction.

These P and N are called a Hahn decomposition of X.
Hahn decompositions are unique up to null sets.

Jordan Decomposition

Definition: *Let (X, A) be a measurable space with two measures ν_1, ν_2. These two measures are mutually singular, denoted by $\nu_1 \perp \nu_2$ if there are measurable sets A, B such that*

$$X = A \cup B, A \cap B = \emptyset, \nu_1(A) = \nu_2(B) = 0$$

Theorem (Jordan Decomposition)**.** *Given a sign measure ν there is a unique pair of measures ν^-, ν^+ such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.*

Proof. Take a Hahn decomposition for ν, call it $X = P \cup N$, then

$$\nu^+(E) = \nu(P \cap E) \quad \nu^-(E) = \nu(N \cap E)$$

ν^+ is called the positive part or the positive variation. ν^- is called the negative part or negative variation.

$$|\nu|(E) = \nu^+(E) + \nu^-(E)$$

Gives a measure which we call the absolute variation.

Lemma.

$$|\nu|(E) = sup\left\{\sum |\nu(E_k)| : \textit{finite collections of pairwise disjoint covers of } E\right\}$$

Lebesgue Decomposition

Definition: *A signed measure ν on a measure space (X, A, μ) is absolutely continuous wrt μ, $\nu \ll \mu$, iff*

$$\forall E \in A, \mu(E) = 0 \implies \nu(E) = 0$$

Lemma. *If*

$$\forall \epsilon > 0 \exists \delta > 0 \ \forall E \in A \ \mu(E) < \delta \implies |\nu(E)| < \epsilon \tag{2.1}$$

then $\nu \ll \mu$. Moreover if $|\nu(X)| < \infty$ then this is iff.

Proof. WLOG we can assume that ν is a measure (using the Jordan decomposition). Now assume

$$\forall \epsilon > 0 \exists \delta > 0 \ \forall E \in A \ \mu(E) < \delta \implies |\nu(E)| < \epsilon$$

Let E be such that $\mu(E) = 0$, then immediately $|\nu(E)| < \epsilon$ for all ϵ, because $\mu(E) < \delta$ for all δ. Hence $|\nu(E)| = 0$.

Converse: Let $|\nu|(X) < \infty$ and assume that $\nu \ll \mu$. Assume for a contradiction that 2.1 does not hold. Then there is some $\epsilon > 0$ and a sequence of measurable sets $\{E_n\}$ such that $\mu(E_n) < 2^{-n}$ but $\nu(E_n) \geq \epsilon$. Let

$$F_n = \bigcup_{k \geq n} E_k \quad F = \bigcap_{n \in \mathbb{N}} F_n$$

Then $F_n \searrow F$ and

$$\mu(F_n) \leq \sum_{k \geq n} \mu(E_k) \leq \sum_{k \geq n} 2^{-k} = 2^{1-n}$$

And by continuity of measure

$$\mu(F) = \lim \mu(F_n) = 0$$

Hence by $\nu \ll \mu$ we get that $\nu(F) = 0$ but by finiteness of ν we also have that

$$\nu(F) = \lim_{n \to \infty} \nu(F_n) \geq \liminf_{n \to \infty} \nu(E_n) \geq \epsilon$$

A contradiction.

Lemma. *Let (X, A, μ) a finite measure space with a finite measure $\nu \ll \mu$. Then there is a measurable function $f : X \to [0, \infty]$ such that*

$$\int_X f d\mu > 0 \quad \wedge \quad \forall E \in A \int_E f d\mu \leq \nu(E)$$

proof on whiteboard, maybe write the key idea down again

Theorem (Radon-Nikodym)**.** *Given a sigma finite measure space (X, A, μ) with a signed measure ν such that $|\nu|(X) < \infty$ and $\nu \ll \mu$ then there is a measurable function $f : X \to \mathbb{R}$ such that*

$$\forall E \in A \;\; \nu(E) = \int_E f d\mu$$

Moreover f is unique (up to a set of measure zero).

proof on whiteboard, maybe write the key idea down again

Theorem (Lebesgue Decomposition)**.** *Given a sigma finite measure space (X, A, μ) with a signed measure ν such that $|\nu|$ is also sigma finite then there is a unique pair of signed measures ν_{ac}, ν_{sing} such that*

$$\nu_{ac} \ll \mu, \;\; |\nu_{sing}| \perp \mu, \;\; \nu = \nu_{ac} + \nu_{sing}$$

Proof. Again assume WLOG that ν is a measure. Let $\lambda = \mu + \nu$ and observe that

$$\int g d\lambda = \int g d\mu + \int g d\nu$$

for g a non-negative measurable function (immediate for simple functions and extend). λ is sigma finite and $\mu \ll \lambda$. Applying Radon-Nikodym gives a non-negative measurable function f such that

$$\mu(E) = \int_E f d\lambda$$

Let $X_+ = \{x : f(x) > 0\}$ and $X_0 = X \setminus X_+$. Let $\nu_+(E) = \nu(E \cap X_+)$ and $\nu_0(E) = \nu(E \cap X_0)$. Then

$$\nu = \nu_0 + \nu_+$$

$\nu_0 \perp \mu$:

$$\mu(X_0) = \int_{X_0} f d\lambda = 0$$

$$\nu(X^+) = \nu(X_0 \cap X_+) = \nu(\emptyset) = 0$$

$\nu_+ \ll \mu$: Let $\mu(E) = 0$ then $\int_E f d\mu = 0$ hence

$$\int_E d\nu = \int_{E \cap X_0} f d\nu + \int_{E \cap X_+} f d\nu = \int_{E \cap X_+} f d\nu = \mu(E) = 0$$

Thus since $f > 0$ on X_+ we must have that $\nu(E) = \nu(E \cap X_+) = 0$.

Uniqueness : Return if I feel.

uniqueness

Differentiating Measures

Covering Lemmas

Lemma (Three Times Covering)**.** *Let $\mathcal{B}$ be a finite collection of closed (or open) balls in $\mathbb{R}^d$, there exists a pairwise disjoint subcollection $\mathcal{B}'$ such that*

$$\bigcup_{B\in\mathcal{B}} B \subseteq \bigcup_{B'\in\mathcal{B}'} 3B'$$

where $3B_r(y) = B_{3r}(y)$

Proof. We sequentially pick the balls with largest radius, then largest radius non-intersecting etc to get

$$\mathcal{B}' = \{B_{r_1}(x_1), ..., B_{r_N}(x_N)\}$$

A collection of pairwise disjoint sets such that $r_1 > r_2 \geq ... \geq r_N$.

Now we want to show that given an arbitrary $B_r(x) \in \mathcal{B} \setminus \mathcal{B}'$ that it is in the three times enlargment of one of the $\mathcal{B}'$.

Because $B_r(x) \notin \mathcal{B}'$ there must be a ball in $\mathcal{B}'$ with a larger radius, moreover there must be a smallest such i.e. $\exists i \;\; r_{i+1} \leq r \leq r_i$ (or i = N). Notice that $\exists j \leq i \;\; B_r(x) \cap B_{r_j}(x_j) \neq \emptyset$ (otherwise $B_r(x) \in \mathcal{B}'$). So let $z \in B_r(x) \cap B_{r_j}(x_j)$ and $y \in B_r(x)$ arbitrary then

$$\begin{aligned} |y - x_j| &\leq |y - x| + |x - x_j| \\ &\leq |y - x| + |x - z| + |x - j - z| \\ &\leq r + r + r_j \leq 3r_j \end{aligned}$$

Hence $B_r(x) \subseteq B_{3r_j}(x_j)$

Lemma (Five Times Covering)**.** *Let $\mathcal{B}$ be a finite collection of closed (or open) balls in $\mathbb{R}^d$, such that*

$$sup\{r : B_r(y) \in \mathcal{B}\} < \infty$$

there exists a pairwise disjoint subcollection $\mathcal{B}'$ such that

$$\bigcup_{B\in\mathcal{B}} B \subseteq \bigcup_{B'\in\mathcal{B}'} 5B'$$

Proof. Let $R = sup\{r : B_r(y) \in \mathcal{B}\}$. For $k \in \mathbb{N}$ define

$$\mathcal{B}_k = \{B \in \mathcal{B} : 2^{-k}R < rad(B) \leq 2^{1-k}R\}$$

Inductively define $\mathcal{B}'_k$ as the maximally pairwise disjoint subset of $\mathcal{B}_k$ that is also pairwise disjoint from $\mathcal{B}'_j$ for $j < k$ (such a set exists by the axiom of choice). Now define

$$\mathcal{B}' = \bigcup_k \mathcal{B}'_k$$

Now take $B = B_r(y) \in \mathcal{B}$ arbitrary. Then $B \in \mathcal{B}_k$ for some k and by maximality of $\mathcal{B}'_k$ we have that there is some $B' = B_{r'}(y') \in \cup_{j=1}^k \mathcal{B}'_j$ such that

$$B \cap B' \neq \emptyset$$

(if none intersected we could add B to $\mathcal{B}'_k$ and contradict maximality). We also see that by definition

$$r \leq 2^{1-k}R = 2 \cdot 2^{-k}R \leq 2r'$$

Now using the argument from the three times covering lemma and $r \leq 2r'$ we are done.

A Vitali covering of $E \in \mathbb{R}^d$ is a collection of closed balls $\mathcal{B}$ such that for every $x \in E$ and every $\delta > 0$ there is a ball $B_r(y) \in \mathcal{B}$ such that $x \in B + r(y)$ and $r < \delta$.

Lemma (Vitali Covering)**.** *Given a set $E \subseteq \mathbb{R}^d$ of finite Lebesgue measure and a Vitali covering of E, $\mathcal{B}$ then there is a countable collection of pairwise disjoint balls $\{B_k\}_{k\geq 1} \subseteq \mathcal{B}$ such tht*

$$m\left(E \setminus \bigcup_{k\in\mathbb{N}} B_k\right) = 0$$

proof is essentially the same as the others I guess, went through it on paper

Lemma (Besicovitch Covering)**.** *Let $\mathcal{B}$ be a collection of balls in $\mathbb{R}^d$ and E the set of all centers of those balls. If*

$$sup\{r : B_r(e) \in \mathcal{B}\} < \infty$$

then there are finitely many subcollections $\mathcal{B}_1, ..., \mathcal{B}_N \subseteq \mathcal{B}$ such that for each j $\mathcal{B}_j$ is a pairwise disjoint collection of balls and

$$E \subseteq \bigcup_{j=1}^{N} \bigcup_{B\in\mathcal{B}_j} B$$

Differentiation of Measures

Given a borel measure μ on $\mathbb{R}^d$ that is finite on all compact sets we definie the upper an lower derivatives of μ at $x \in \mathbb{R}^d$ respectively as

$$\bar{D}\mu(x) = \limsup_{r\to 0^+} \frac{\mu(B_r(x))}{m(B_r(x))}$$

$$\underline{D}\mu(x) = \liminf_{r\to 0^+} \frac{\mu(B_r(x))}{m(B_r(x))}$$

If both are finite and equal then we call them the derivative, $D\mu(x)$ or $\frac{d\mu}{dm}$, of μ at x.

Lemma. *$\bar{D}\mu(x)$ and $\underline{D}\mu(x)$ are Borel measurable.*

Proof.

Upper: For every $\delta > 0$ we define

$$s_\delta(x) = \sup_{0<r<\delta} \frac{\mu(B_r(x))}{m(B_r(x))}$$

Notice that $\bar{D}\mu(x)$ is the limit as $\delta \to 0$ hence if we can show this function is measurable we are done.

Let $a > 0$ (WLOG because the other cases are trivial). We will show that $\{x : s_\delta(x) > a\}$ is open. So let $x \in \{x : s_\delta(x) > a\}$

$$s_\delta(x) = \sup_{0<r<\delta} \frac{\mu(B_r(x))}{m(B_r(x))} > a$$

$$\implies \exists a' > a, t \in (0, \delta) \quad \frac{\mu(B_t(x))}{m(B_t(x))} > a$$

Now let $\rho > 0$ such that $(\frac{t}{t+\rho})^d a' > a$ and $\rho + t < \delta$. If $y \in B_\rho(x)$ then by monotonicity (and triangle inequality)

$$\mu(B_{t+\rho}(y)) \geq \mu(B_t(x)) > a'm(B_t(x)) = a'(\frac{t}{t+\rho})^d m(B_{t+\rho}(y)) > am(B_{t+\rho}(y))$$

(equality is from Lebesgue measure independence of location and then scaling the ball). Hence

$$\frac{\mu(B_{t+\rho}(y))}{m(B_{t+\rho}(y))} > a$$

For every such t, ρ hence the sup, so $s_\delta(y) > a$ for every $y \in B_\rho(x)$. Thus the set is open and hence measurable.

Lemma. *If $E \subseteq \mathbb{R}^d$ Borel such that $\forall x \in E \;\; \bar{D}\mu(x) \geq a > 0$ then $\mu(E) \geq am(E)$*

Proof. Let E be Borel such that $\bar{D}\mu(E) \geq a$. WLOG assume E is bounded (finite measures). Take an $0 < \epsilon < a$ and a U open such that $E \subseteq U$ and $\mu(E) < \mu(E) + \epsilon$. Define

$$\mathcal{B} = \{B = B_r(x) \subseteq U : \mu(B) \geq (a - \epsilon)m(B)\}$$

Since $\bar{D}\mu(E) \geq a$ $\mathcal{B}$ forms a Vitalli covering of E.

thats going to take a bit more convincing

Apply the Vitali covering Lemma to get a pairwise disjoint subcollection $\{B_k\}_{k\in\mathbb{N}}$ that cover E up to a set of measure 0. Then

$$(a-\epsilon)m(E) \leq \sum_k (a-\epsilon)m(B_k) \leq \sum_k \mu(B_k) \leq \mu(U) \leq \mu(E) + \epsilon \to \mu(E)$$

proof is similar to the previous

Lemma. *If $\mu \ll m$ and $E \subseteq \mathbb{R}^d$ Borel such that $\forall x \in E$ $\underline{D}\mu(x) \leq a > 0$ then $\mu(E) \leq am(E)$*

Theorem. *μ is differentiable m-a.e. and for any E Borel there is a measure zero Borel set Z such that*

$$\mu(E) = \int_E D\mu(x)dm(x) + \mu(E \cap Z)$$

proof

Proof. WLOG assume $\mu \ll m$ and that μ is finite. Given $a < b \in \mathbb{R}$ define

$$F_{a,b} = \{x : \underline{D}\mu(x) < a < b < \bar{D}\mu(x)\}$$

$$F_\infty = \{x : \bar{D}\mu(x) = \infty\}$$

By the previous lemmas we then have

$$m(F_\infty) \leq \frac{1}{a}\mu(F_\infty)$$

for all $a > 0$ and so $m(F_\infty) < \infty$ (by finitness of μ).

A Borel measure μ satisfies the Vitali property if for any finite measure E and every Vitali covering $\mathcal{B}$ of E there is a countable subcollection $\{B_i\} \subseteq \mathcal{B}$ of pairwise disjointn balls such that

$$\mu\left(E \setminus \bigcup_{k\in\mathbb{N}} B_k\right) = 0$$

Theorem. *Given two Borel measures that are finite on compact sets μ, ν such that ν satisfies the Vitali property we have that μ is differentiable wrt ν, i.e.*

$$\frac{d\mu}{d\nu}(x) = \lim_{r\to0^+} \frac{\mu(B_r(x))}{\nu(B_r(x))}$$

exists ν-a.e..

Moreover for any E Borel there is a ν-measure zero Borel set Z such that

$$\mu(E) = \int_E \frac{d\mu}{d\nu}(x)d\nu(x) + \mu(E \cap Z)$$

Finally if μ satisfies the Vitali property too then $Z = \{x : \frac{d\mu}{d\nu}(x) = \infty\}$.

Anti-Derivatives

Given an integrable function $f : \mathbb{R}^d \to \mathbb{R}$ we call $x \in \mathbb{R}^d$ a Lebesgue point of f iff

$$\lim_{r\to0^+} m(B_r(x))^{-1} \int_{B_r(x)} |f(y) - f(x)|dm(y) = 0$$

The Lebesgue set is the collection of all such points.

Theorem. *For such an f we have that a.e. x is a Lebesgue point and*

$$\lim_{r\to 0^+} m(B_r(x))^{-1}\int_{B_r(x)} f(y)dm(y) = f(x)$$

proof

Lemma. *If A is Lebesgue measurable then*

$$\lim_{r\to 0^+} \frac{m(A\cap B_r(x))}{m(B_r(x))} = 1 \quad a.e.\ x\in A$$

$$\lim_{r\to 0^+} \frac{m(A\cap B_r(x))}{m(B_r(x))} = 0 \quad a.e.\ x\in \mathbb{R}^d$$

proof

Todo list